DETERMINING THE THERMAL STRESSES IN A HOLLOW VISCOELASTIC SPHERE

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For a viscoelastic sphere with spherical cavity, solution of the quasistatic problem of the stresses produced by a nonstationary temperature field reduces to solution of an integral–differential equation whose right side depends on an unknown function of the time. A numerical solution method is described.

We are to compute the stresses and strains in a viscoelastic sphere of radius R with a spherical cavity of radius r_0 with specified stresses on the boundaries of the region, and specified stresses at the initial time t_0:

$$\sigma_r|_{r=r_0} = \sigma_r(r_0, t),\ \sigma_r|_{r=R} = \sigma_r(R, t),\ \sigma_r|_{t=t_0} = \sigma_r(r, t_0) \tag{1}$$

for a linear law of viscoelasticity [2]:

$$\left(q_0 + q_1 \frac{\partial}{\partial t}\right)\frac{\partial \varepsilon_\varphi}{\partial r} + \left(p_0 + p_1 \frac{\partial}{\partial t}\right)\frac{\partial \sigma_r}{\partial r} = 0, \tag{2}$$

$$e = \frac{1-2\mu}{E} s + \alpha T. \tag{3}$$

Here

$$T|_{r=r_0} = T(r_0, t),\ T|_{r=R} = T(R, t),\ T|_{t=t_0} = T(r, t_0), \tag{4}$$

$$\varepsilon_\varphi|_{\substack{r=r_0,\\ t=t_0}} = \varepsilon_\varphi(r_0, t_0). \tag{5}$$

The coefficients $q_0(T)$, $q_1(T)$, $p_0(T)$, $p_1(T)$ depend arbitrarily on the temperature T.

For spherical symmetry, the complete system of equations for the linear quasistatic viscoelastic problem [1, 2] consists of the equilibrium equation

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r}(\sigma_r + \sigma_\varphi) = 0, \tag{6}$$

the consistency condition

$$\varepsilon_r = r\frac{\partial \varepsilon_\varphi}{\partial r} + \varepsilon_\varphi, \tag{7}$$

the heat-conduction equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{a}\frac{\partial T}{\partial t} \tag{8}$$

and the law of viscoelasticity (2), (3) (all equations are represented in dimensionless variables).

The initial and boundary conditions are given only for σ_r, so that we reduce the system (2), (6) to an integral–differential equation containing the second mixed derivative with respect to σ_r. Using (3), (7), we eliminate σ_φ from (6). We divide the resulting equation by $K = E/(1-2\mu)$, and integrate from r_0 to r; then

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$$\varepsilon_\varphi = \frac{\sigma_r}{K} + \frac{1}{r^3}\int_{r_0}^{r} r^2\left(r\frac{\sigma_r}{K^2}\frac{\partial K}{\partial r} + 3\alpha T\right)dr - \frac{C_1}{r^3},$$
$$C_1 = \left[r^3\left(\frac{\sigma_r}{K} - \varepsilon_\varphi\right)\right]\Big|_{r=r_0}. \tag{9}$$

Substituting ε_φ into (2), we obtain the following equation for σ_r:

$$\frac{\partial^2\sigma_r}{\partial r\partial t} = A(r, t),$$

$$A(r, t) = \frac{K}{q_1 + p_1K}\left\{q_1\left[\left(\frac{3\sigma_r}{r} + U\right)\frac{\partial}{\partial t}\left(\frac{1}{K}\right) - \frac{3}{r}\frac{\partial(\alpha T)}{\partial t}\right] - \frac{3T}{r}q_0\alpha\right.$$
$$- \left(p_0 + \frac{q_0}{K}\right)U + \frac{3q_0}{r^4}\int_{r_0}^{r} r^2\left(\frac{r}{K}\frac{\partial K}{\partial r}\sigma_r + 3\alpha T\right)dr$$
$$\left. + \frac{3q_1}{r^4}\int_{r_0}^{r} r^2\left[\frac{r}{K}\frac{\partial K}{\partial r}V + 3\frac{\partial(\alpha T)}{\partial t} - (3\sigma_r + rU)\frac{1}{K^2}\frac{\partial K}{\partial t}\right]dr + \frac{3}{r^4}(q_0C_2 + q_1C_3)\right\}; \tag{10}$$
$$C_2 = \left[r^3\left(\varepsilon_\varphi - \frac{\sigma_r}{K}\right)\right]\Big|_{r=r_0}; \quad C_3 = \left[r^3\left(\frac{\partial\varepsilon_\varphi}{\partial t} - \frac{V}{K}\right)\right]\Big|_{r=r_0};$$
$$U = \frac{\partial\sigma_r}{\partial r}, \quad V = \frac{\partial\sigma_r}{\partial t}.$$

The arbitrary time function $\varepsilon_\varphi(r_0, t) = f(t)$ and its derivative $\partial\varepsilon_\varphi(r_0, t)/\partial t = f'(t)$ occur in C_2, C_3.

To solve this problem, we solve (10) so as to satisfy the initial condition (1). After we have found σ_r, we can determine the remaining stress and strain components from (6), (7), (9).

The lines r = const, t = const are characteristics of (10).

In contrast to the ordinary Gursat problem, where the initial data are known for two characteristics, in our case the initial conditions are specified on three characteristics, $t = t_0$, $r = r_0$, $r = R$, but the function $f(t)$ occurring in the coefficient of (10) is arbitrary. It must be so defined that all three conditions on the characteristics are satisfied.

The system (10), (8) with given conditions (1), (4), (5) cannot be solved analytically when $T \not\equiv$ const. When numerical methods are employed, it is essential to establish the correctness of the given problem. This is not difficult to do for a model equation with constant coefficients,

$$\frac{\partial^2\sigma_r}{\partial r\partial t} + a\frac{\partial\sigma_r}{\partial r} - \frac{f(t)}{r^4} = 0,$$

which is obtained from (10) when $T \equiv$ const. It is easy to obtain an exact solution for this equation; its form shows the correctness of the problem as formulated.

Let us now describe a numerical method for solving (10), which is based on the method of characteristics [3].

We construct a rectangular net, formed by the characteristics $r = r_0 + ih$, $t = t_0 + j\tau$ ($i = 0, 1, \ldots, n$; $j = 0, 1, \ldots, m$) of (10). The equations of the characteristics and the differential relationships along them are as follows:

Family I	Family II
$dr = 0,$	$dt = 0,$
$dt - AdU = 0,$	$dr - AdV = 0,$
$d\sigma_r = Vdt,$	$d\sigma_r = Udr,$

(11)

while (8) is replaced by approximating difference equations with order of approximation $O(h + \tau)$. The difference equation for heat conduction is solved by the "dispersion" method [4], i.e., we can assume that the temperature is specified within the region.

The final computational formulas for (10) are

$$U_{i,j} = A_{i,j-1}\tau + U_{i,j-1}, \tag{12}$$

$$V_{i,j} = A_{i-1,j}h + V_{i-1,j}, \tag{13}$$

$$\sigma_{i,j} = \frac{1}{2}\left[\sigma_{i,j-1} + \sigma_{i-1,j} + U_{i-1,j}h + V_{i,j-1}\tau\right], \tag{14}$$

where

$$\begin{aligned} A_{i,j}\left(\frac{q_{1i,j} + p_{1i,j}K_{i,j}}{K_{i,j}}\right) &= \frac{q_{1i,j}}{K_{i,j}^2}\left(\frac{3\sigma_{i,j}}{r_i} + U_{i,j}\right)\Delta_2 K - \left(p_{0i,j} + \frac{q_{0i,j}}{K_{i,j}}\right)U_{i,j} - \frac{3T_{i,j}}{r_i}\left(q_{0i,j}\alpha_{i,j} + q_{1i,j}\Delta_2\alpha\right) \\ &+ q_{1i,j}S\left[\frac{r^3}{K}V\Delta_1 K + 3r^2(T\Delta_2\alpha + \alpha\Delta_2 T)\right] + q_{0i,j}S\left(\frac{r^3}{K^2}\sigma\Delta_1 K + 3r^2\alpha T\right) \\ &- \frac{3}{2}\frac{q_{1i,j}h}{r_i^4}\sum_{k=1}^{i}\left[\frac{r_k^2}{K_{k,j}^2}(3\sigma_{k,j} + r_k U_{k,j})\Delta_{-1}K\right] - \frac{3q_{1i,j}\alpha_{i,j}}{r_i}\Delta_2 T \\ &+ \frac{3r_0^3}{r_{i,j}^4}\left[q_{0i,j}\left(f_j - \frac{\sigma_{0,j}}{K_{0,j}}\right) + q_{1i,j}\left(\frac{f_{j+1} - f_j}{\tau} - \frac{V_{0,j}}{K_{0,j}}\right)\right]; \end{aligned} \tag{15}$$

$$\Delta_1\varphi = \frac{\varphi_{i+1,j} - \varphi_{i,j}}{h}, \quad \Delta_2\varphi = \frac{\varphi_{i,j+1} - \varphi_{i,j}}{\tau}, \quad \Delta_{-1}\varphi = \frac{\varphi_{i,j} - \varphi_{i-1,j}}{\tau};$$

$$S(\psi) = \frac{3h}{2r_i^4}\sum_{k=1}^{i}(\psi_{k,j} - \psi_{k-1,j}).$$

If for the j-th series we know $\sigma_{i,j}$, $U_{i,j}$, f_j, then $V_{i,j}$ can be found from (13), with the solution being refined in accordance with the following formula [3]:

$$V_{i,j} = \frac{h}{2}(A_{i-1,j} + A_{i,j}) + V_{i-1,j}, \tag{16}$$

as soon as we determine the value of $f_{j+1}(t)$ that occurs in $A_{i,j}$.

Letting $f_{j+1} \equiv 0$, we find $V_{i,j}$ from (13), (16) with a certain error $M_{i,j}f_{j+1}$. If we trace the increase in the error from point to point, we can obtain an expression for $M_{i,j}$,

$$\begin{aligned} M_{i,j} &= \left[1 + \frac{h}{2}(a_{i-1,j} + a_{i,j}) + \frac{h^2}{2}a_{i-1,j}a_{i,j}\right]M_{i-1,j} \\ &+ hc_{i,j}\sum_{k=1}^{i-1}d_{k,j}M_{k,j} + \frac{h}{2}(1 + ha_{i,j})b_{i-1,j} + \frac{h}{2}b_{i,j} + hc_{i-1,j}(1 + ha_{i,j})\sum_{k=1}^{i-2}d_{k,j}M_{k,j}, \end{aligned} \tag{17}$$

where

$$a_{i,j} = \frac{3q_{1i,j}h\Delta_1 K}{2r_i K_{i,j}(q_{1i,j} + p_{1i,j}K_{i,j})}; \quad c_{i,j} = \frac{3q_{1i,j}r_0^3 K_{i,j}}{r_i^4(q_{1i,j} + K_{i,j}p_{1i,j})\tau};$$

$$b_{i,j} = \frac{2r_0^3}{\tau}c_{i,j}; \quad d_{i,j} = \frac{r_i^4 h\Delta_1 K}{K_{i,j}^2}, \quad i \geqslant 1$$

(when i = 1, 2, the corresponding sums in (17) drop out).

As a result of determining $V_{i,j}$, we obtain a certain value $V_{n,j}$ at the point (R, t_j). On the other hand, from (1) we can find the exact value of $V(R, t_j)$, after which

$$f_{j+1} = \frac{V(R, t_j) - V_{n,j}}{M_{n,j}}. \tag{18}$$

Knowing the proper value of f_{j+1}, we recompute $V_{i,j}$ and proceed to compute $\sigma_r(r, t)$ and $U(r, t)$ for the (j + 1)-st series. No additional iterations are required to determine $V_{i,j}$.

We compute $U_{i,j+1}$ from (12) and then use the formula

$$U_{i,j+1} = \frac{1}{2}(A_{i,j} + A_{i,j+1})\tau + U_{i,j} \tag{19}$$

to refine the solution. The unknown value $V_{i,j+1}$ occurs in $A_{i,j+1}$, however. Thus the function $U(r, t)$ is refined by iteration. We first let $V_{i,j+1} = V_{i,j}$ and compute $\overset{0}{U}_{i,j+1}$ for the entire series. We next find $V_{i,j+1}$ and compute $\overset{1}{U}_{i,j+1}$, etc. The iteration process terminates when

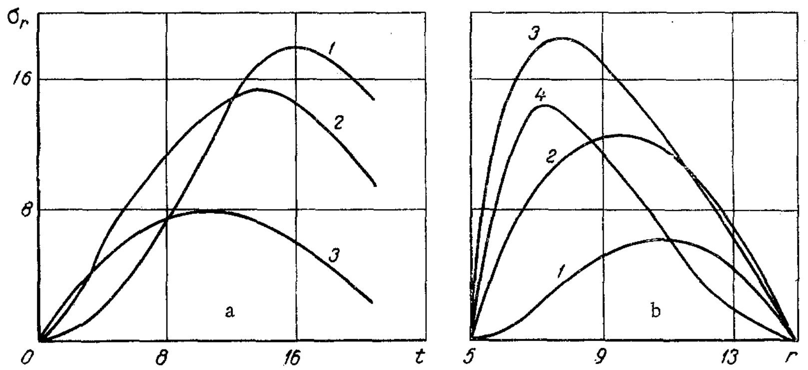

Fig. 1. Stresses σ_r as function of time t and of radius r (a, b, respectively): a) 1) r = 7; 2) 10; 3) 13; b) 1) t = 4; 2) 9; 3) 16; 4) 21.

$$|\overset{l}{U}_{i,j+1} - \overset{l-1}{U}_{i,j+1}| < \delta, \tag{20}$$

where l is the number of the iteration, and δ is the specified error.

We compute $\sigma_{i,j+1}$ at the same time as $U_{i,j+1}$. The series is first evaluated by means of (14), and in subsequent calculations the series $\sigma_{i,j+1}$ is refined from the following formula [5]:

$$\sigma_{i,j+1} = \sigma_{i,j} + \sigma_{i-1,j+1} + \frac{h}{2}(\overset{l}{U}_{i-1,j+1} + \overset{l}{U}_{i,j+1}) + \frac{\tau}{2}(V_{i,j} + \overset{l}{V}_{i,j+1}). \tag{21}$$

Writing (6), (9), (7) in terms of differences, we can obtain formulas for σ_φ, ε_φ, ε_r.

The method was tested in the class of sufficiently smooth solutions for point solutions obtained when $T \equiv \text{const}$ and $\varepsilon_\varphi = \sigma_r$.

Evaluation of different versions indicated the stability of the proposed computational scheme. It also turned out that one iteration was usually sufficient to satisfy (20).

As an example, we calculated the stresses in a hollow sphere ($r_0 = 5$, $R = 15$) of epoxy resin [6], for which

$$\eta = 10443\cdot\exp(-0.0275T),\ \alpha = 8\cdot 10^{-5},$$
$$E = -1.75T + 352.5,\ \mu = 0.4,$$
$$p_0 = \frac{1}{\eta},\ p_1 = 1,\ q_0 = 0,\ q_1 = \frac{E}{1+\mu}.$$

The calculations were carried out under the following conditions:

$$T|_{t=0} = 36;\ T|_{r=5} = \begin{cases} 45 - \frac{1}{4}(t-6)^2, & 0 \leqslant t < 14, \\ 30 & , 14 \leqslant t \leqslant 21; \end{cases}$$

$$T|_{r=15} = 72 - \frac{1}{4}(t-12)^2,\ \varepsilon_\varphi|_{\substack{r=5,\\ t=0}} = 0;$$

$$\sigma_r|_{t=0} = \sigma_r|_{r=5} = \sigma_r|_{r=15} = 0.$$

The solution results are shown graphically in Fig. 1.

The method proposed can be employed effectively to design structures of the hollow-sphere type made from viscoelastic materials with arbitrary temperature characteristics and an arbitrarily varying temperature field.

NOTATION

σ_r, σ_φ are the normal stresses at areas with normals r, φ;
s is the average normal stress;
ε_r, ε_φ are the radial and circumferential strains;

e	is the average elongation;
r	is the radius of the sphere;
t	is the time;
$T(r, t)$	is the temperature;
$q_0(T)$, $q_1(T)$, $p_0(T)$, $p_1(T)$	are the parameters of viscoelasticity;
α	is the coefficient of thermal expansion;
μ	is the Poisson ratio;
E	is the Young's modulus;
K	is the bulk modulus;
η	is the viscosity;
h, τ	are the radius and time steps;
$\sigma_{i,j}$, $U_{i,j}$, $V_{i,j}$, $T_{i,j}$, $A_{i,j}$, $q_{0i,j}$, $q_{1i,j}$, $p_{0i,j}$, $p_{1i,j}$, $\alpha_{i,j}$, $K_{i,j}$	are the values of the corresponding functions at the point $r_i = r_0 + ih$, $t_j = t_0 + j\tau$.

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